Centre de recherche sur les transports (C.R.T.)
Centre for Research on Transportation

CRT-2003-15

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June 2003

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103-07110/

PUBLICATION
CRT-2003-15

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* This research was supported in part by the Dirección de Investigactión, Universidad de Conception, the Natural Sciences and Engineering Research Council of Canada (NSERC), MIDEPLAN-Chile, Fondap Matemáticas Aplicadas and CONICYT-Chile.

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Dépôt légal – Bibliothèque nationale du Québec
Bibliothèque nationale du Canada, 2003

Abstract

This paper concerns the route choice on congested transit networks. The issue addressed is how to model the consequences of congestion on the predicted flows and on the expected waiting and travel times. The paper builds on the results presented in [7, 5], extending these to formulate a solvable model for large scale transit networks. This new model formulation is a generalization of the previously known strategy (hyperpath) based transit network equilibrium models, and leads to an algorithm which has been applied successfully on large scale networks. Computational results for transit networks originating from practice demonstrate the applicability of the proposed approach.

Key words: transit assignment, capacity constraints, gap function.

Résumé

Ce papier concerne le problème de sélection d'itinéraire dans les réseaux de transport en commun congestionnés. La question adressée est celle de la modélisation des effets de la congestion sur les flots ainsi que sur les temps d'attente et de voyage espérés. Le papier s'inspire des résultats présentés dans [7, 5], en les étendant pour obtenir un modèle résoluble pour des réseaux de transport public de grande taille. Cette nouvelle formulation est une généralisation des modèles d'équilibre de réseaux basés sur la notion de stratégie (hyperchemin), et conduit à un algorithme qui a été appliqué avec succès pour des réseaux de grande taille. Des résultats numériques sur des réseaux de transport en commun issus de la pratique démontrent l'applicabilité de l'approche proposée.

Mots clefs : affectation transport en commun, contraintes de capacité, fonction de gap.

1 Introduction

The methods used for planning urban and interurban transit services rely on the use of transit assignment models which predict the way that transit travelers choose routes from their origins to their destinations. While much progress has been achieved in the past two decades in the development of models for transit route choice, the issue of how to model the consequences of congestion on the predicted flows and expected travel times for large scale networks received relatively little attention. The purpose of this paper is to build on the results of Correa [7] and Cominetti and Correa [5], and extend these to the formulation of a solvable model for general large scale transit networks. This topic was the subject of the doctoral thesis of Cepeda [3].

It is worthwhile to review the contributions made to the study of transit route choice. The earliest algorithms suggested to find paths in a transit network — such as Dial [12], Fearnside and Draper [13] and LeClerq [18] — recognized that waiting time at stops served by several lines was an important aspect of this problem, and proposed various heuristic ways to combine waiting and travel times in computing a shortest path in the transit network. The seminal paper of Chriqui and Robillard [4] introduced the notion that, on a simple network of one origin and one destination, passengers can select a subset of lines and board the first one of these that arrives at a stop in order to minimize the expected waiting and travel times. The results of Chriqui and Robillard were extended to general transit networks in two ways. Spiess [20] and Spiess and Florian [21] introduced the notion of strategy, which is a choice of an attractive set of lines at each decision point in a transit itinerary on a general transit network; that is, at each node where boarding occurs. The resulting model and algorithm achieve the minimization of the expected value of the total travel time which includes access, wait and in-vehicle time. Nguyen and Pallottino [19] provided a graph theoretic interpretation of a strategy as an acyclic directed graph for each destination, and denoted it as a hyperpath. These models may be extended to consider congestion aboard the vehicles by associating monotone discomfort functions with each segment of a transit line. The resulting transit equilibrium models may be solved by standard algorithms for convex cost minimization problems. However, since they consider that the frequencies are fixed, the waiting times are underestimated since in a period of heavy congestion passengers may not be able to board the first vehicle to arrive at a stop.

The results of Chriqui and Robillard were used in a different way by DeCea and Fernandez [9] in a transit assignment model which considers minimal routes. Their model only considers choices among multiple lines that serve a stop only if they all share the next stop to be served. It can be shown that the optimal routes found by this model is a subset of the optimal strategy (yet for one particular network this difference was not significant — see [8]). DeCea and Fernandez [10] extended the minimal route model to consider increased waiting times at stops and congestion aboard the vehicles. Their approach leads to an asymmetric equilibrium model which is solved by the Jacobi method. The computational effort is significant and the resulting flows often exceed the capacity of the vehicles since the waiting time functions resemble the volume/delay functions used in the modeling of car networks.

Other contributions to this problem include the work of Gendreau [15] who was the first to study the congestion effects at bus stops and to formulate a general route choice model that considered congestion. Wu, Florian and Marcotte [22] attempted to model the increased waiting times by augmenting the time required to board a vehicle, without changing the nominal frequencies. More recently, Bouzaiene-Ayari [1] and later Bouzaiene-Ayari, Gendreau and Nguyen [2] extended the strategy route choice model to consider increased waiting times at stops due to congestion. The distribution of transit flows among attractive lines is

done in inverse proportion of the waiting times, which may be flow dependent. However, the congestion functions used preclude the use of waiting time functions which tend to infinity as the transit flows reach the capacity of the vehicles. An empirical study reported by Harris [16] describes the calibration of a congestion function for the London underground and its use in a transit planning suite for the planning of capacity expansion.

More recently, Cominetti and Correa [5] considered the common lines problem of Chriqui and Robillard [4] and formulated a new model where line capacities are considered and the waiting times at the stop node increase as the capacity of the vehicle is reached. These waiting time functions, or equivalently line frequency functions, are motivated by queueing theory and tend to infinity as the flows reach capacity. An extension to a general network is presented as well. The purpose of this contribution is to extend this model to a general network and arrive to a model formulation which is a generalization of the previously known strategy based transit network equilibrium models. This formulation leads to an algorithm which is used successfully for large scale networks.

The paper is organized as follows. The common lines problem is revisited in the next section and the extension to a general network is accomplished in section 3. Some special cases of the model are discussed in sections 4 and 5. The solution algorithm, as well as some comments on the difficulty of the problem and numerical results obtained with the algorithm are given in section 6. A short conclusion is provided at the end.

2 The congested common-line problem revisited

Let us recall the common-line problem under congestion described in [5]. Consider a simple network consisting of an origin O connected to a destination D by a finite set of bus lines $A=\{a_1,\ldots,a_n\}$. Each line $a \in A$ is characterized by a constant in-vehicle travel time $t_a \in \mathbb{R}$ and an *effective frequency* function $f_a : [0,\bar{v}_a) \to (0,\infty)$ which is assumed to be a smooth function of the line flow v_a with $f_a'(\cdot) < 0$ and $f_a(v_a) \to 0$ when $v_a \to \bar{v}_a$. The effective frequency is supposed to be decreasing in order to reflect the increment in waiting time induced by an augmentation of flow. The constant $\bar{v}_a > 0$ (eventually $\bar{v}_a = \infty$) is called the *saturation flow* of the line.

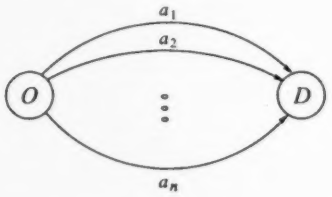

Figure 1: The common-line problem

For the purpose of traveling from O to D, each passenger is assumed to select a nonempty subset of lines $s \subset A$, called the *attractive lines* or simply a *strategy*, boarding the first incoming bus from this set with available capacity. In this way the total flow $x \geq 0$ splits among all possible strategies $s \in \mathcal{S}$ so that $x = \sum_{s\in\mathcal{S}} h_s$ where $h_s \geq 0$ denotes the flow on strategy $s \in \mathcal{S}$. Assuming that a passenger using strategy s will end up boarding line $a \in s$ with probability $\pi_a^s = f_a(v_a)/\sum_{b\in s} f_b(v_b)$, it turns out that each strategy-flow vector $h = (h_s)_{s\in\mathcal{S}}$ induces a unique vector of line-flows $v = v(h)$ through the system of equations[1]

$$(E) \qquad v_a = \sum_{s\in\mathcal{S}} h_s \frac{f_a(v_a)}{\sum_{b\in s} f_b(v_b)} \qquad \forall\, a \in A,$$

which determine in turn the expected transit time of each strategy

$$(T) \qquad T_s(v) \triangleq \frac{1+\sum_{a\in s} t_a f_a(v_a)}{\sum_{a\in s} f_a(v_a)}.$$

Invoking Wardrop's principle, a strategy-flow vector $h \geq 0$ with $\sum_{s\in\mathcal{S}} h_s = x$ is said to be an *equilib-*

[1] Existence of a solution for this system requires in fact a non-saturation condition on h (see [5]).

rium iff all strategies carrying flow are of minimal time, that is to say,

$$(W) \qquad h_s > 0 \;\Rightarrow T_s(v(h)) = \tau(v(h))$$

where $\tau(v) \triangleq \min_{s\in\mathcal{S}} T_s(v)$.

Denote by H_x the set of all equilibrium h's. It was shown in [5] that H_x may be fully characterized by an equivalent optimization problem expressed directly in terms of the line-flows v. In particular, denoting V_x the set of line-flows $v(h)$ corresponding to all possible equilibria $h \in H_x$, it turns out (see [5, Theorem 1.2]) that there exists a constant $\alpha_x \geq 0$ such that $v \in V_x$ if and only if $v \geq 0$ with $\sum_{a\in A} v_a = x$ and

$$(C_0) \qquad \frac{v_a}{f_a(v_a)} \begin{cases} = \alpha_x & \text{if } t_a < \tau(\hat{v}(\alpha_x)) \\ \leq \alpha_x & \text{if } t_a = \tau(\hat{v}(\alpha_x)) \\ = 0 & \text{if } t_a > \tau(\hat{v}(\alpha_x)) \end{cases}$$

where $\hat{v}_a(\alpha)$ denotes the unique solution z of the equation $z/f_a(z) = \alpha$.

Our first result provides a simpler characterization of V_x. Formally the condition looks very similar to (C_0) but it avoids the functions $\hat{v}_a(\alpha)$. Notice also that the α_x which is common to all $v \in V_x$ is now allowed to change for each $v \in V_x$ (although in fact it will not!). These are slight but crucial differences which permit to obtain an equivalent optimization problem characterizing the network equilibrium as described in the next section. The alternative characterization of V_x is as follows.

Theorem 2.1 *$v \in V_x$ if and only if $v \geq 0$ with $\sum_{a\in A} v_a = x$ and there exists $\alpha \geq 0$ such that*

$$(C_1) \qquad \frac{v_a}{f_a(v_a)} \begin{cases} = \alpha & \text{if } t_a < \tau(v) \\ \leq \alpha & \text{if } t_a = \tau(v) \\ = 0 & \text{if } t_a > \tau(v). \end{cases}$$

The proof exploits the following characterization of optimal strategies [5, Lemma 1.2 and Cor. 1.1].

Lemma 2.1 *Let $\check{s}(v) \triangleq \{a : t_a < \tau(v)\}$ and $\hat{s}(v) \triangleq \{a : t_a \leq \tau(v)\}$. Then the following are equivalent:*
(a) $T_s(v) = \tau(v)$,
(b) $t_a \leq T_s(v) \leq t_b \quad \forall\, a \in s, b \notin s$,
(c) $\check{s}(v) \subseteq s \subseteq \hat{s}(v)$.

Proof of Theorem 2.1.
($\Rightarrow$) Let $v \in V_x$ — so that (C_0) holds — and take $\alpha = \alpha_x$. According to [5, Theorem 1.2] one has $\tau(v) = \tau(\hat{v}(\alpha_x))$, so that condition (C_1) follows at once from (C_0).

($\Leftarrow$) To prove the converse, let us assume that the vector v satisfies (C_1) for some $\alpha \geq 0$. It will be shown that $\tau(v) = \tau(\hat{v}(\alpha))$. Indeed, let $s = \check{s}(v)$ so that Lemma 2.1 implies that $\tau(v) = T_s(v)$ and $t_a \leq T_s(v) \leq t_b$ for all $a \in s$, $b \notin s$. Now, from (C_1) one gets $v_a = \hat{v}_a(\alpha)$ for all $a \in s$ and since $T_s(\cdot)$ depends only on these flows it follows that $T_s(v) = T_s(\hat{v}(\alpha))$. Hence $t_a \leq T_s(\hat{v}(\alpha)) \leq t_b$ for all $a \in s$, $b \notin s$, and using Lemma 2.1 once again one gets $T_s(\hat{v}(\alpha)) = \tau(\hat{v}(\alpha))$. Putting together these equalities it follows that $\tau(v) = \tau(\hat{v}(\alpha))$ as claimed. Now, it was already noticed that (C_1) implies $v_a = \hat{v}_a(\alpha)$ if $t_a < \tau(v) = \tau(\hat{v}(\alpha))$, and similarly one obtains $v_a \leq \hat{v}_a(\alpha)$ if $t_a = \tau(\hat{v}(\alpha))$ and $v_a = 0$ if $t_a > \tau(\hat{v}(\alpha))$. These estimates, combined with $\sum_{a\in A} v_a = x$, imply

$$\sum_{a\in\check{s}(\hat{v}(\alpha))} v_a \;\leq\; x \;\leq\; \sum_{a\in\hat{s}(\hat{v}(\alpha))} v_a$$

which according to [5, Theorem 1.2] yields $\alpha = \alpha_x$ and $v \in V_x$ as was to be proved. ■

2.1 Dealing with infinite frequencies

When modeling general transit networks many arcs have no waiting time and are assigned infinite frequencies (see next section). In order to extend the previous model, it suffices to adopt the convention that all computations are performed by first replacing the infinite frequencies by a common finite value f which is then considered to tend to ∞. In particular, if a strategy s contains one or more arcs with infinite frequencies $s_\infty \subseteq s$, the transfer time $T_s(v)$ turns out to be the average of the travel times t_a for $a \in s_\infty$; while the boarding probabilities are $\pi_a^s = 1/|s_\infty|$ for $a \in s_\infty$ and $\pi_a^s = 0$ otherwise. Theorem 2.1 still holds in this more general setting, provided that the third case of condition (C_1) is interpreted as "$v_a = 0$ if $t_a > \tau(v)$". For more details we refer to [3, Section 3.6].

3 The network equilibrium model

The transit network is represented by a directed graph $G=(N,A)$ which is built as follows (see [21]). Let $N_s \subset N$ be a set of nodes representing the bus stops in the network. Each bus line l is represented by a set of line-nodes $N_l \subset N$ which correspond to the sequence of bus stops served by the line. Each line-node in N_l connects to the corresponding stop-node in N_s through boarding and alight arcs, as well as to the next line-node in the sequence through an on-board arc (also called line segment). Eventually, one may consider walk arcs connecting directly a pair of nodes in N_s.

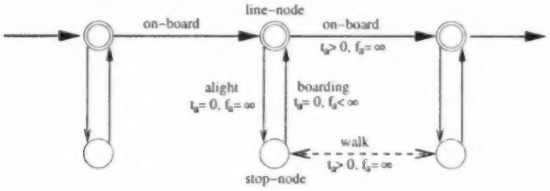

Figure 2: Representation of a transit network

The set of destinations is denoted $D \subseteq N$, and for each $d \in D$ and every node $i \neq d$ a fixed demand $g_i^d \geq 0$ is given. In general the demands g_i^d are strictly positive only at nodes i corresponding to stop-nodes, that is to say the bus stops where users wait for service. Also, each arc $a \in A$ has an associated travel-time $t_a : \mathcal{V} \to [0,\infty)$ and an effective frequency function $f_a : \mathcal{V} \to [0,\infty]$ where $\mathcal{V} = [0,\infty)^{A\times D}$ denotes the space of destination-arc-flow vectors v with non-negative entries $v_a^d \geq 0$ for all $a \in A, d \in D$. Typically, one assigns infinite effective frequencies $f_a(\cdot) \equiv \infty$ to all the arcs except the boarding arcs (where the waiting processes occur), while for travel times one takes $t_a(\cdot) \equiv 0$ on all arcs except for the on-board and walk arcs which are assigned a positive travel time function $t_a(\cdot) > 0$. However, one may also assign a positive travel time to the alight and boarding arcs, in order to represent the times spent on embark/disembark operations or the effect of the bus fare on the user choices.

The set of feasible flows $\mathcal{V}_0$ is defined as the set of all vectors $v \in \mathcal{V}$ such that $v_a^d = 0$ for all $a \in A_d^+$ (*i.e.* no flow with destination d exits from d) and satisfying the flow conservation constraints

$$g_i^d + \sum_{a\in A_i^-} v_a^d = \sum_{a\in A_i^+} v_a^d \qquad \forall\, i \neq d. \quad (1)$$

Here $A_i^+ = \{a : i_a = i\}$ and $A_i^- = \{a : j_a = i\}$ are the forward and backward stars of node $i \in N$, with i_a and j_a denoting respectively the tail and head nodes of arc $a \in A$.

3.1 The notion of network equilibrium

The intuitive idea behind the notion of a network equilibrium is the following. Consider a passenger heading towards destination d and reaching an intermediate node i in his trip. To exit from i he can use the arcs $a \in A_i^+$ to reach the next node j_a. By taking the arc travel times $t_a(v)$ and the transit times $\tau_{j_a}^d$ from j_a to d as given, the decision faced at node i is a common-line problem with travel times $t_a(v) + \tau_{j_a}^d$ and effective frequencies corresponding to the services operating on the arcs $a \in A_i^+$. The solution of this common-line problem determines the transit time τ_i^d from i to d, which can then be used recursively to solve the up-stream nodes. It is important to stress that every node i must be considered as a potential origin even if the initial demand g_i^d is zero, since it may receive transfer flows coming from other nodes and heading towards d.

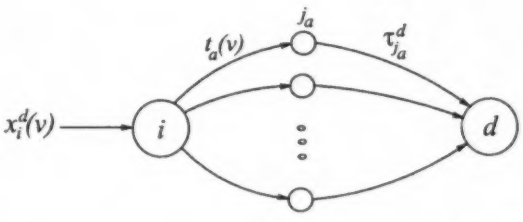

Figure 3: The i-to-d common-line problem

Since all the variables τ_i^d and v_a^d must be determined at the same time, the transit network model

is stated as a set of simultaneous common-line problems (one for each pair i, d) coupled by flow conservation constraints. More precisely, for each $v \in \mathcal{V}$ the flow entering node i with destination d is defined as $x_i^d(v) \triangleq g_i^d + \sum_{a \in A_i^-} v_a^d$, while the *time-to-destination* functions $\tau_i^d(v)$ are the unique solution of the generalized Bellman equations [19, 21, 5]:

$$\begin{cases} \tau_d^d = 0 \\ \tau_i^d = \min\limits_{s \in \mathcal{S}_i} \dfrac{1 + \sum_{a \in s} [t_a(v) + \tau_{j_a}^d] f_a(v)}{\sum_{a \in s} f_a(v)}, \quad \forall\, i \neq d, \end{cases}$$

where $\mathcal{S}_i$ stands for the set of nonempty subsets $s \subseteq A_i^+$. The quantity $t_a^d(v) \triangleq t_a(v) + \tau_{j_a}^d(v)$ represents the minimal time to destination d when using arc a. Let $V_i^d(v)$ denote the set of equilibrium flows corresponding to a common-line problem defined by the arcs A_i^+, with total flow $x_i^d(v)$, constant travel times $t_a^d(v)$, and diagonal effective frequencies $f_a^d(\cdot)$ obtained from $f_a(\cdot)$ by considering it as a function of the flow v_a^d alone, and keeping the other variables fixed.

Definition 3.1 *A feasible flow $v \in \mathcal{V}_0$ is called a* network equilibrium *iff for all $d \in D$ and $i \neq d$ the flows $(v_a^d)_{a \in A_i^+}$ belong to $V_i^d(v)$. The set of network equilibrium flows will be denoted $\mathcal{V}_0^*$.*

By using the strategy-flow variables h_s^d one may restate the above definition in the following terms. A vector $v \in \mathcal{V}$ is an equilibrium iff for all $d \in D$ and $i \neq d$ there exist $\tau_i^d \in \mathbb{R}$ and strategy-flows $h_s^d \geq 0$ for $s \in \mathcal{S}_i$ satisfying the flow equations

$$\sum_{s \in \mathcal{S}_i} h_s^d = x_i^d(v)$$

$$v_a^d = \sum_{s \ni a} h_s^d \frac{f_a(v)}{\sum_{b \in s} f_b(v)} \qquad \forall\, a \in A_i^+,$$

and the Wardrop's equilibrium conditions

$$T_s^d(v) \begin{cases} = \tau_i^d & \text{if} \quad h_s^d > 0 \\ \geq \tau_i^d & \text{if} \quad h_s^d = 0, \end{cases}$$

where the cost of strategy $s \in \mathcal{S}_i$ is given by

$$T_s^d(v) = \frac{1 + \sum_{a \in s} [t_a(v) + \tau_{j_a}^d] f_a(v)}{\sum_{a \in s} f_a(v)}.$$

REMARK. Notice the fact that the solution set $\mathcal{V}_0^*$ is made up only of vectors such that $f_a(v) > 0$. Although the functions $f_a(\cdot)$ are defined over all the space $\mathcal{V}$, the effective frequency $f_a(v)$ is positive only if the flow v strictly respects the capacity of all the arcs (*i.e.* if some arcs are saturated then there must also be arcs with zero effective frequency). Consequently, the space $\mathcal{V}_0$ could be redefined to include the constraints $\sum_{d \in D} v_a^d < \bar{v}_a$, where $\bar{v}_a$ is the capacity of arc a. In practice, one can think that only the line segments have a finite capacity, which is equal to the nominal frequency of the corresponding bus line μ_l, multiplied by the bus capacity K_l.

3.2 Characterization of the equilibrium

In the sequel it will be assumed that the set of network equilibrium flows $\mathcal{V}_0^*$ is nonempty (for sufficient conditions see [5, Theorem 2.1]). As a direct application of the results of section 2, a network equilibrium may be characterized as follows.

Theorem 3.1 *$v \in \mathcal{V}_0^*$ iff $v \in \mathcal{V}_0$ and there exist numbers $\alpha_i^d \geq 0$ such that for all $d \in D$ and $i \neq d$*

$$\frac{v_a^d}{f_a(v)} \begin{cases} = \alpha_i^d & \text{if} \quad t_a^d(v) < \tau_i^d(v) \\ \leq \alpha_i^d & \text{if} \quad t_a^d(v) = \tau_i^d(v) \\ = 0 & \text{if} \quad t_a^d(v) > \tau_i^d(v). \end{cases}$$

where $t_a^d(v) \triangleq t_a(v) + \tau_{j_a}^d(v)$.

This characterization allows to derive a gap function with the property that each of its minimizers corresponds precisely to a network equilibrium.

Theorem 3.2 *For all $v \in \mathcal{V}_0$, $d \in D$ and $i \neq d$ the inequality*

$$\sum_{a \in A_i^+} [t_a(v) + \tau_{j_a}^d(v)] v_a^d + \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} \geq \tau_i^d(v) \sum_{a \in A_i^+} v_a^d \qquad (2)$$

holds. Moreover $v \in \mathcal{V}_0^*$ iff $v \in \mathcal{V}_0$ and all these inequalities are satisfied as equalities.

Proof. Although using the convention in 2.1 the proof given below holds for infinite frequencies, the reader may assume for simplicity that all the functions $f_a(v)$ are finite. Moreover, the functional dependencies of t_a, f_a, τ_i^d and t_a^d on the vector v are omitted, and Δ is written for the expression on the left of (2).

Fix $v \in \mathcal{V}_0$, $d \in D$, $i \neq d$, and consider the vector $(h_s^d)_{s\in\mathcal{S}_i}$ computed by the algorithm

$$(A)\quad \left\{\begin{array}{l} w_a^d \leftarrow v_a^d \text{ for all } a \in A_i^+ \\ h_s^d \leftarrow 0 \text{ for all } s \in \mathcal{S}_i; \\ \textbf{while } \text{“}s \triangleq \{a \in A_i^+ : w_a^d > 0\} \neq \phi\text{”} \\ \quad h_s^d \leftarrow \min\{w_a^d[\textstyle\sum_{b\in s} f_b]/f_a : a \in s\} \\ \quad w_a^d \leftarrow w_a^d - h_s^d f_a/[\textstyle\sum_{b\in s} f_b] \text{ for all } a \in s \\ \textbf{end}. \end{array}\right.$$

This iteration decomposes v_a^d for $a \in A_i^+$ as

$$v_a^d = \sum_{s\ni a} h_s^d \frac{f_a}{\sum_{b\in s} f_b} \tag{3}$$

and since $\sum_{a\in A_i^+}\sum_{s\ni a} \equiv \sum_{s\in\mathcal{S}_i}\sum_{a\in s}$ one gets

$$\sum_{a\in A_i^+} v_a^d = \sum_{s\in\mathcal{S}_i} h_s^d.$$

Moreover, it is easy to check that v_a^d/f_a is maximal iff a belongs to every s generated by (A) (*i.e.* all s such that $h_s^d > 0$), in which case the sum on the right hand side of (3) may be extended to all $s \in \mathcal{S}_i$ and therefore

$$\max_{a\in A_i^+} \frac{v_a^d}{f_a} = \sum_{s\in\mathcal{S}_i} \frac{h_s^d}{\sum_{b\in s} f_b}. \tag{4}$$

By replacing (3) and (4) into the expression for Δ and exchanging the order of summation it results that

$$\begin{aligned} \Delta &= \sum_{s\in\mathcal{S}_i} h_s^d \frac{1+\sum_{a\in s}[t_a+\tau_{j_a}^d]f_a}{\sum_{a\in s} f_a} \\ &\geq \sum_{s\in\mathcal{S}_i} h_s^d \tau_i^d = \tau_i^d \sum_{a\in A_i^+} v_a^d \end{aligned}$$

hence proving (2). These inequalities become equalities iff for all s such that $h_s^d > 0$ one has

$$\frac{1+\sum_{a\in s}[t_a+\tau_{j_a}^d]f_a}{\sum_{a\in s} f_a} = \tau_i^d,$$

that is to say, if all the strategies s generated by (A) are optimal. By Lemma 2.1 this is equivalent to the fact that s must contain every arc such that $t_a^d < \tau_i^d$ and no arc with $t_a^d > \tau_i^d$. By a previous observation, this corresponds exactly to the condition that v_a^d/f_a is maximal if $t_a^d < \tau_i^d$ and 0 if $t_a^d > \tau_i^d$, and then according to Theorem 3.1 it follows that (2) hold as equalities iff v is a network equilibrium. ∎

The result above shows that the equilibrium flows $v \in \mathcal{V}_0^*$ are the global minima of the function

$$\sum_{\substack{d\in D\\ i\neq d}}\left[\sum_{a\in A_i^+}[t_a(v)+\tau_{j_a}^d(v)]v_a^d + \max_{a\in A_i^+}\frac{v_a^d}{f_a(v)} - \tau_i^d(v)\sum_{a\in A_i^+} v_a^d\right]$$

with optimal value equal to 0. A straightforward computation permits to state the problem of finding equilibrium flows in the form

$$(P)\qquad \underset{v\in\mathcal{V}_0}{\text{Minimize}}\ G(v)$$

where

$$G(v) \triangleq \sum_{d\in D}\left[\sum_{a\in A} t_a(v)v_a^d + \sum_{i\neq d}\max_{a\in A_i^+}\frac{v_a^d}{f_a(v)} - \sum_{i\neq d} g_i^d \tau_i^d(v)\right].$$

4 Global vs. local optimality

Although problem (P) characterizes the network equilibria, if we attempt to solve it by using a descent method, the iterates may be trapped into a local minimum which is not an equilibrium. To avoid such pathological situation it would suffice to establish that all local minima of (P) are global. It is unlikely that such a result holds in general, but it is possible to prove it for the special case

of common-lines, namely a 2-node network linked by finitely many arcs $a \in A$ with constant travel times $t_a(v) = t_a$ and decreasing diagonal effective frequencies $f_a(v) = f_a(v_a)$. As a matter of fact, in this case one can show that every stationary point is automatically an equilibrium and hence a global optimum. In order to prove the latter, let us consider a common-line problem with total demand $x > 0$ so that (P) becomes

$$(\tilde{P}) \quad \min_{v} \sum_{a\in A} t_a v_a + \max_{a\in A} \frac{v_a}{f_a(v_a)} - x \min_{s\in\mathcal{S}} T_s(v)$$
$$\text{s.t. } v_a \geq 0 \; ; \; \textstyle\sum_{a\in A} v_a = x$$

with

$$T_s(v) = \frac{1+\sum_{a\in s} t_a f_a(v_a)}{\sum_{a\in s} f_a(v_a)}.$$

Let us rewrite this problem as

$$\begin{aligned}(\tilde{P}) \quad & \min_{(v,\alpha,\tau)} \sum_{a\in A} t_a v_a + \alpha - x\tau \\ & \text{s.t.} \\ & 0 \leq v_a \leq \alpha f_a(v_a), \; \forall a \in A \\ & \textstyle\sum_{a\in A} v_a = x \\ & \tau \leq T_s(v), \; \forall s \in \mathcal{S}\end{aligned}$$

and introduce multipliers $\mu \in \mathbb{R}$, $\beta_a \geq 0$, $\gamma_a \geq 0$ and $h_s \geq 0$ in order to build the Lagrangian

$$\mathcal{L} \triangleq \sum_{a\in A} t_a v_a + \alpha - x\tau + \mu[x - \sum_{a\in A} v_a] - \sum_{a\in A} \beta_a v_a + \sum_{a\in A} \gamma_a[v_a - \alpha f_a(v_a)] + \sum_{s\in\mathcal{S}} h_s[\tau - T_s(v)].$$

A feasible point (v, α, τ) for (P) is a KKT point if there exist multipliers (μ, β, γ, h) satisfying the complementarity conditions

$$h_s > 0 \;\Rightarrow\; T_s(v) = \tau \tag{5}$$
$$\beta_a > 0 \;\Rightarrow\; v_a = 0 \tag{6}$$
$$\gamma_a > 0 \;\Rightarrow\; v_a = \alpha f_a(v_a), \tag{7}$$

and the stationarity conditions $\frac{\partial \mathcal{L}}{\partial \tau} = 0$, $\frac{\partial \mathcal{L}}{\partial \alpha} = 0$ and $\frac{\partial \mathcal{L}}{\partial v_a} = 0$, which correspond to

$$\sum_{s\in\mathcal{S}} h_s = x \tag{8}$$

$$\sum_{a\in A} \gamma_a f_a(v_a) = 1 \tag{9}$$

$$t_a - \mu - \beta_a + \gamma_a[1 - \alpha f_a'(v_a)] = \sum_{s\ni a} \frac{h_s}{\sum_{b\in s} f_b(v_b)}[t_a - T_s(v)] f_a'(v_a). \tag{10}$$

To simplify the notation, the dependence of f_a and f_a' on v_a is omitted. Also set $\eta = \mu - \tau$ and

$$\alpha_a = \sum_{s\ni a} \frac{h_s}{\sum_{b\in s} f_b(v_b)}.$$

Then, taking into account (5), equation (10) may be rewritten as

$$(t_a - \tau)[1 - \alpha_a f_a'] = \beta_a - \gamma_a[1 - \alpha f_a'] + \eta. \tag{11}$$

Proposition 4.1 *If (v, α, τ) is a KKT point for $(\tilde{P})$ then v is a common-line equilibrium.*

Proof. The constraints of (P) imply $\tau \leq \min_s T_s(v)$. Now, from (8) it follows that there exists $s \in \mathcal{S}$ with $h_s > 0$ so that (5) implies $T_s(v) = \tau$ and consequently $\tau = \min_{s\in\mathcal{S}} T_s(v)$. The result will follow from Theorem 2.1 if it is shown that $\eta = 0$, since in this case (11) combined with the complementarity conditions imply

$$\begin{cases} t_a < \tau & \Rightarrow \quad \gamma_a > 0 \quad \Rightarrow \quad v_a/f_a = \alpha \\ t_a > \tau & \Rightarrow \quad \beta_a > 0 \quad \Rightarrow \quad v_a/f_a = 0. \end{cases}$$

In order to show that $\eta = 0$ note that β_a and γ_a may not be positive simultaneously (otherwise one would have $\alpha = 0$, so that $v_a = 0$ for all a and then $x = 0$). Also, every $s \in \mathcal{S}$ with $h_s > 0$ is such that $T_s(v) = \tau$, hence $T_s(v)$ is minimal and according to Lemma 2.1 s contains all the arcs such that $t_a < \tau$ and no arc with $t_a > \tau$. It follows that

$$\begin{cases} t_a < \tau & \Rightarrow \quad \alpha_a = \bar{\alpha} \\ t_a > \tau & \Rightarrow \quad \alpha_a = 0, \end{cases}$$

with $\bar{\alpha} = \sum_{s\in\mathcal{S}} \frac{h_s}{\sum_{b\in s} f_b(v_b)}$. Finally, remark that

$$\sum_{a\in A} \alpha_a f_a \;=\; \sum_{a\in A} \sum_{s\ni a} \frac{h_s}{\sum_{b\in s} f_b} f_a$$

$$= \sum_{s\in\mathcal{S}}\sum_{a\in s}\frac{h_s f_a}{\sum_{b\in s} f_b}$$
$$= \sum_{s\in\mathcal{S}} h_s = x = \sum_{a\in A} v_a$$

which gives

$$0 = \sum_{a\in A}\left[\alpha_a - \frac{v_a}{f_a}\right] f_a. \qquad (12)$$

Let us show that $\eta = 0$ by contradiction.

Suppose first $\eta < 0$ and let $s_0 = \{a : v_a > 0\}$. For $a \in s_0$ the value of β_a equals 0 and then (11) yields $t_a < \tau$ and $\gamma_a < (\tau - t_a)[1 - \bar{\alpha} f'_a]/[1 - \alpha f'_a]$. Also, by noting that $\gamma_a = 0$ for $a \notin s_0$ (since $\gamma_a > 0$ implies $v_a = \alpha f_a > 0$), and by using (9) one gets

$$1 = \sum_{a\in s_0}\gamma_a f_a < \sum_{a\in s_0}(\tau - t_a) f_a \frac{[1 - \bar{\alpha} f'_a]}{[1 - \alpha f'_a]}. \qquad (13)$$

Now, $\tau \le T_{s_0}(v)$ implies $\sum_{a\in s_0}(\tau - t_a) f_a \le 1$, which combined with (13) leads to

$$0 < \sum_{a\in s_0}(\tau - t_a) f_a \left(\frac{[1 - \bar{\alpha} f'_a]}{[1 - \alpha f'_a]} - 1\right),$$

or equivalently

$$0 < (\alpha - \bar{\alpha}) \sum_{a\in s_0}\frac{(\tau - t_a) f_a f'_a}{[1 - \alpha f'_a]}.$$

Since each term in this last sum is negative it follows that $(\alpha - \bar{\alpha}) < 0$. By substituting this back into (12) and by using the fact that $\alpha_a = \bar{\alpha}$ for $a \in s_0$ and $v_a = 0$ for $a \notin s_0$, the following contradiction is obtained

$$0 = \sum_{a\in s_0}[\bar{\alpha} - \frac{v_a}{f_a}] f_a + \sum_{a\notin s_0}\alpha_a f_a \ge (\bar{\alpha} - \alpha)\sum_{a\in s_0} f_a > 0.$$

Suppose now that $\eta > 0$ and let $s_1 = \{a : t_a \le \tau\}$. For $a \in s_1$ the equation (11) implies that $\gamma_a > 0$, and therefore one has $v_a/f_a = \alpha$, $\beta_a = 0$ as well as $\gamma_a > (\tau - t_a)[1 - \bar{\alpha} f'_a]/[1 - \alpha f'_a]$. These conditions, combined with (9) give

$$1 \ge \sum_{a\in s_1}\gamma_a f_a > \sum_{a\in s_1}(\tau - t_a) f_a \frac{[1 - \bar{\alpha} f'_a]}{[1 - \alpha f'_a]}.$$

By Lemma 2.1 we have $\tau = T_{s_1}(v)$ which implies $\sum_{a\in s_1}(\tau - t_a) f_a = 1$. This leads to

$$0 > \sum_{a\in s_1}(\tau - t_a) f_a \left(\frac{[1 - \bar{\alpha} f'_a]}{[1 - \alpha f'_a]} - 1\right)$$

and further

$$0 > (\alpha - \bar{\alpha}) \sum_{a\in s_1}\frac{(\tau - t_a) f_a f'_a}{[1 - \alpha f'_a]}$$

which yields $(\alpha - \bar{\alpha}) > 0$. Going back to (12) once again and by using now the fact that $v_a/f_a = \alpha$ for $a \in s_1$ and $\alpha_a = 0$ for $a \notin s_1$, one reaches the contradiction

$$0 = \sum_{a\in s_1}[\alpha_a - \alpha] f_a - \sum_{a\notin s_1} v_a \le (\bar{\alpha} - \alpha)\sum_{a\in s_1} f_a < 0.$$

These two contradictions prove that $\eta = 0$, thus completing the proof. ■

Corollary 4.1 *If (v, α, τ) is a local minimum for $(\tilde{P})$ then it is a global minimum and v is a common-line equilibrium.*

Proof. It is easy to check that (v, α, τ) satisfies the Mangasarian-Fromovitz constraint qualification so that it is a KKT point. Proposition 4.1 then implies that v is an equilibrium, and then Theorem 2.1 implies that v is a global optimum with cost equal to zero. ■

The next section goes back to the case of general networks, showing the relation between problem (P) and two previous strategy-based transit assignment models: the linear cost and the convex cost models of Spiess [20] and Nguyen and Pallotino [19]. The linear cost model serves as a building block in the solution algorithm proposed for problem (P) in section 6 of the paper.

5 Two special cases

Two special cases of formulation (P) are particularly interesting, as they correspond exactly to the models studied by Spiess [20] as well as Nguyen and Pallottino [19]. The first one, called the uncongested case, is characterized by constant travel times $t_a(v) \equiv t_a$ and constant effective frequencies $f_a(v) \equiv f_a$. The second case, called semi-congested, only takes the functions $f_a(v) \equiv f_a$ to be constant, whereas the functions $t_a(v)$ are used to model the congestion effects. In both cases problem (P) becomes

$$\underset{v \in \mathcal{V}_0}{\text{Min}} \sum_{d \in D} \left[\sum_{a \in A} t_a(v) v_a^d + \sum_{i \neq d} \max_{a \in A_i^+} \frac{v_a^d}{f_a} - \sum_{i \neq d} g_i^d \tau_i^d(v) \right]$$

which can be rewritten as

$$(P_1) \quad \underset{v \in \mathcal{V}_0}{\text{Min}} \sum_{d \in D} \left[\sum_{a \in A} t_a(v) v_a^d + \sum_{i \neq d} w_i^d - \sum_{i \neq d} g_i^d \tau_i^d(v) \right]$$

$$\text{s.t.} \quad v_a^d \leq w_i^d f_a \quad \forall\, d \in D, i \neq d, a \in A_i^+.$$

It is clear that for any optimal solution of the latter and any pair (i, d), the constraints $v_a^d \leq w_i^d f_a$ must be satisfied as equality for at least one $a \in A_i^+$; since otherwise, it would be possible to reduce the value of the objective function by reducing the corresponding variable w_i^d.

5.1 The uncongested case

In the uncongested case the functions $\tau_i^d(v)$ become constant, and so does the third term in the objective function. Hence problem (P_1) reduces exactly to the linear program found by Spiess [20], where w_i^d represents the waiting time at node i of all passengers going to destination d. In this setting, the time-to-destination variables τ_i^d correspond exactly to the dual variables of the linear programming formulation, and the third term of the objective function in problem (P_1) is the objective value of the dual so that the condition

$$\sum_{d \in D} \left[\sum_{a \in A} t_a v_a^d + \sum_{i \neq d} w_i^d - \sum_{i \neq d} g_i^d \tau_i^d \right] = 0$$

is simply a consequence of the strong duality theorem of linear programming.

5.2 The semi-congested case

Consider now the case where congestion is introduced only through segment crowding functions in such a way that the cost of arc a is an increasing function of the total flow $v_a = \sum_{d \in D} v_a^d$ on that arc. As shown in Spiess [20], the transit equilibrium may be found as the solution of the following equivalent convex cost minimization problem

$$(PC) \quad \underset{v \in \mathcal{V}_0}{\text{Min}} \sum_{a \in A} \int_0^{v_a} t_a(x)dx + \sum_{\substack{d \in D \\ i \neq d}} w_i^d$$

$$\text{s.t.} \quad v_a^d \leq w_i^d f_a \qquad d \in D, i \neq d, a \in A_i^+$$

$$v_a = \textstyle\sum_{d \in D} v_a^d \quad a \in A.$$

The first order optimality conditions for this convex cost problem are

$$\sum_{a \in A} t_a(v_a)(v_a' - v_a) + \sum_{\substack{d \in D \\ i \neq d}} [w_i^d(v') - w_i^d(v)] \geq 0$$

for all $v' \in \mathcal{V}_0$, where $w_i^d(v) = \max_{a \in A_i^+} v_a^d / f_a$ and $v_a = \sum_{d \in D} v_a^d$. An optimal solution of (PC) satisfies

$$\sum_{a \in A} t_a(v_a) v_a + \sum_{\substack{d \in D \\ i \neq d}} w_i^d(v) - \sum_{\substack{d \in D \\ i \neq d}} g_i^d \tau_i^d(v) = 0$$

and therefore the problem can be reformulated as

$$\underset{v \in \mathcal{V}_0}{\text{Min}} \sum_{d \in D} \left[\sum_{a \in A} t_a(v_a) v_a^d + \sum_{i \neq d} \max_{a \in A_i^+} \frac{v_a^d}{f_a} - \sum_{i \neq d} g_i^d \tau_i^d(v) \right]$$

which is another particular case of (P), even though this from is not amenable to efficient computations.

6 A solution algorithm

Some attempts were made to find an exact approach to solve problem (P). Among the algorithms tried was a projected generalized gradient method, similar to the one used by Constantin [6]. This attempt was not successful due to three main factors: (i) it was not possible to find an analytical expression for the projected generalized gradient; (ii) the tests carried out on a small network which permitted an analytical expression of the projection exhibited an extremely slow convergence; and (iii) it seems rather difficult to prove convergence of this method.

From a different perspective, since the relation between strategy-flows and arc-flows is nonlinear, it was not possible to formulate the problem in the form of a variational inequality, which would open the way for using numerical methods for VI's. However, the model can be written as a quasi-variational inequality (see [3, pages 64–66]), but this approach was not pursued in order to develop an algorithm.

Now, since the optimal value of (P) is known, a simple alternative is to use a heuristic method and to evaluate the deviation from optimality of the computed solutions by using the value of the objective function. One of the simplest — yet very robust — schemes for solving equilibrium problems, is the Method of Successive Averages (MSA). At each iteration, the method computes a transit network equilibrium for the linear network obtained by fixing the travel times and the frequencies at the values determined by the current flows, and then updates these flows by averaging the previous iterate and the newly computed solution. As mentioned earlier, an equilibrium for the linear cost network can be found by solving

$$(PL) \quad \min_{v \in \mathcal{V}_0} \sum_{d \in D} \left[\sum_{a \in A} t_a v_a^d + \sum_{i \neq d} w_i^d \right]$$

$$\text{s.t.} \quad v_a^d \leq w_i^d f_a \quad d \in D, i \neq d, a \in A_i^+$$

which amounts to compute a shortest hyperpath for each destination $d \in D$, and then determining the corresponding destination-arc-flow vector. Hence, choosing $\alpha_k \in (0,1)$ with $\alpha_k \to 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$, the MSA can be described as follows:

ALGORITHM MSA

Initialize: find $v^0 \in \mathcal{V}_0$ and set $k \leftarrow 0$

while $G(v^k) > \epsilon$

- compute $t_a = t_a(v^k)$ and $f_a = f_a(v^k)$
- compute shortest hyperpaths for each $d \in D$
- determine the induced flows $\hat{v}_a^d$
- update $v^{k+1} = (1 - \alpha_k)v^k + \alpha_k \hat{v}$
- set $k \leftarrow k + 1$

end

Stop: v^k is a solution with gap $G(v^k) \leq \epsilon$.

The computation of the gap $G(v^k)$ requires the storage of the flow variables v_a^{kd} for every destination $d \in D$ and all arcs $a \in A$ (in fact it suffices to consider the arcs having a finite frequency, *i.e.* the boarding arcs). As the network and the number of destinations grows, the storage requirements grows accordingly but even for large networks this remains within the RAM capacity of standard personal computers.

It is worthwhile to note that if there is insufficient capacity to carry all the demand for travel on the transit network, the application of the MSA algorithm results in a solution that is not feasible. When such a condition occurs, one can allow the pedestrian mode on all the links of the underlying street network, except for transit modes that operate on exclusive tracks such as the underground and the rail modes. Then the MSA algorithm will find an approximate equilibrium solution with large increases in pedestrian flows, which can be viewed as flows on "slack" arcs. The location of these pedestrian flows would indicate the corridors of the transit network that require additional capacity.

6.1 Numerical experiences

The MSA algorithm was implemented as a macro procedure within the EMME/2 software package [17]. The latter provides a module that computes an optimal hyperpath with linear costs and fixed frequencies on a general transit network, and which is used to solve the linearized sub-problems that compute $\hat{v}$ in the algorithm stated above.

The transit networks used for the numerical tests presented below originate from the cities of Winnipeg, Stockholm and Santiago, Chile. The origin-destination matrices correspond to real estimated demands.

6.1.1 Stockholm and Winnipeg

The transit network of the City of Stockholm consists of 6 modes, 6 transit vehicle types, 185 centroids, 395 regular nodes, 2079 directional links, 76 transit lines and 1096 transit line segments.

In the case of the City of Winnipeg, the transit network consists of 154 centroids, 903 regular nodes, 2975 directional links, 133 transit lines and 4338 transit line segments. There are 2 transit vehicle types but only one transit mode, the bus.

Figures 4 and 5 below illustrate the evolution of the first 70 MSA iterations, by plotting an approximation of the *relative gap*, that is to say, the gap $G(v^k)$ expressed as a percentage of the total travel time $\sum_{\substack{d \in D \\ i \neq d}} g_i^d \tau_i^d(v^k)$. In both cases, the convergence towards 0 is relatively fast and smooth, reaching a value close to 0.25% after 70 iterations. A closer look reveals that in the case of Stockholm there is not enough capacity to accomodate the demand and the problem is infeasible. Indeed, the maximum value of the flow-to-capacity ratio over all the line segments, which must be less than 1 if capacities are respected, drops from 4.77 to 1.5 stabilizing at this value after 10 iterations. Simultaneously, the number of over-saturated line segments having a ratio larger than one, reduces from 45 arcs in the first iteration to 4 arcs at iteration 10, oscillating thereafter around this value without ever reaching a feasible flow. This indicates the existence of a bottleneck due to an overestimated demand or because some line capacities are too small. The over-saturated line segments are easily identified, allowing to determine the source of the conflict. This situation does not occur in the case of Winnipeg where the maximun flow-to-capacity ratio drops from 2.12 to 0.97, while the over-saturated transit line segments are reduced from 221 to 0, stabilizing at these feasible values after 9 iterations.

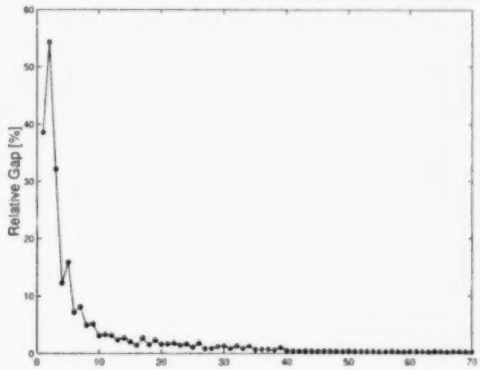

Figure 4: MSA for Stockholm

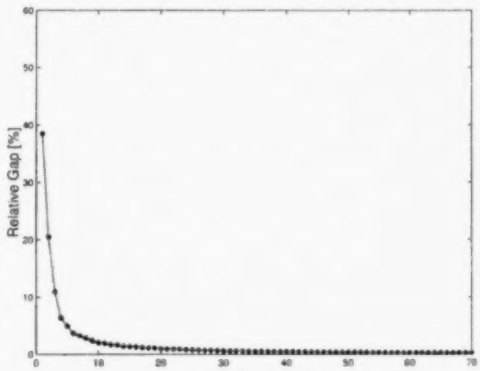

Figure 5: MSA for Winnipeg

6.1.2 The Santiago equilibrium model

The network of the City of Santiago, Chile has 409 centroids, 1808 nodes, 11331 directional links,

1116 transit lines and 52468 transit line segments. There are 11 modes including 5 pure transit modes and 4 combined modes (such as auto-metro, etc.).

The transportation planning model includes all the demand computation steps and the network assignment for 13 socioeconomic classes and 3 trip purposes. The model is quite complex and may be referred to in [11, 14]. The critical part of the transit network was the underground (metro) which was consistently over simulated. The MSA algorithm was embedded in the complex equilibration procedure that solves the model, after which the resulting metro flows satisfied the capacity constraints. The flows on the different segments of metro line 5, before and after equilibration, are illustrated in figure 6 below.

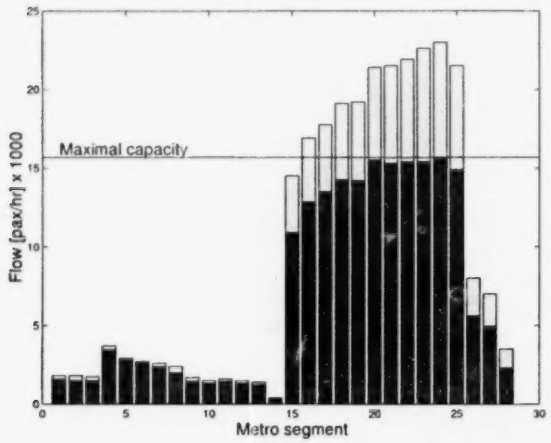

Figure 6: Flows on metro line 5

7 Conclusion

The developments presented in this paper open the way for the use of a new transit assignment model in congested networks where it is important to respect the capacity of the services offered. In addition to the usual urban transit networks used in transportation planning models, the model may be applied to airline services or train services with reserved seats. The theoretical analysis presented provides a solid foundation for a simple and robust algorithm since the nearness to an exact equilibrium solution is easily quantified. The empirical convergence of the algorithm raises the interesting issue of devising a rigorous convergence proof which is still an open research question.

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